Sums of large global solutions to the incompressible Navier-Stokes equations

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SUMS OF LARGE GLOBAL SOLUTIONS TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

JEAN-YVES CHEMIN, ISABELLE GALLAGHER, AND PING ZHANG

Abstract. Let $\mathcal{G}$ be the (open) set of $\dot H^{\frac12}(\mathbb{R}^3)$ divergence free vector fields generating global smooth solutions to the three dimensional incompressible Navier-Stokes equations. We prove that any element of $\mathcal{G}$ can be perturbed by an arbitrarily large, smooth divergence free vector field which varies slowly in one direction, and the resulting vector field (which remains arbitrarily large) is an element of $\mathcal{G}$ if the variation is slow enough. This result implies that through any point in $\mathcal{G}$ passes an uncountable number of arbitrarily long segments included in $\mathcal{G}$.

1. Introduction

1.1. **Setting of the problem and statement of the result.** Let us first recall the classical Navier-Stokes system for incompressible fluids in three space dimensions:

$$(NS) \qquad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \end{cases}$$

where $u(t,x)$ denotes the fluid velocity and $p(t,x)$ the pressure. In this paper the space variable x is chosen in $\mathbb{R}^3$.

All the solutions we are going to consider here are at least continuous in time with values in the Sobolev space $\dot H^{\frac12}(\mathbb{R}^3)$. It is well known that in that case, all concepts of solutions coincide and in particular we shall deal with ″mild" solutions of (NS) (see for instance [16]).

In order to specify the concept of large data, let us recall the history of results concerning small data. The first one states that if the initial data u_0 is such that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ is small enough, (NS) has a global regular solution; this was proved by J. Leray in his seminal paper [17]. Then, starting with the paper by H. Fujita and T. Kato (see [9]), the following approach was developped: let us denote by $\mathbb{B}$ the bilinear operator defined by

$$\begin{cases} \partial_t \mathbb{B}(v,w) - \Delta\mathbb{B}(v,w) = \dfrac12 \mathbb{P}\operatorname{div}(v\otimes w + w\otimes v) \\ \mathbb{B}(v,w)_{|t=0} = 0 \end{cases}$$

where $\mathbb{P}$ denotes the Leray projection onto divergence free vector fields. Then, it is easily checked that u is a solution of (NS) if and only if

$$u = e^{t\Delta}u_0 + \mathbb{B}(u,u)$$

which is something like Duhamel's formula. Then the theory of small initial data reduces to finding a Banach space X of time-dependent divergence free vector fields on $\mathbb{R}^+\times\mathbb{R}^3$ such that $\mathbb{B}$ is a bilinear map from $X\times X$ to X. An elementary abstract fixed point theorem claims that if X is a Banach space of time-dependent divergence free vector fields on $\mathbb{R}^+\times\mathbb{R}^3$ such that

$$\|\mathbb{B}(v,w)\|_X \le C\|v\|_X\|w\|_X$$

(X will be called from now on an adapted space), a solution of (NS) exists in X and is global as soon as
$$\|e^{t\Delta}u_0\|_X \leq (4C)^{-1}.$$
The search of the largest possible adapted space X is a long story. It started in 1964 with the paper [9] where the space X is defined by the norm
$$\|u\|_X \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac14}\|\nabla u(t)\|_{L^2}.$$
This corresponds to an initial data small in $\dot H^{\frac12}(\mathbb{R}^3)$, and it is shown in particular that the solution belongs to $C(\mathbb{R}^+, \dot H^{\frac12}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot H^{\frac32}(\mathbb{R}^3))$. After a number of important steps (see in particular [10], [13], [22] and [3]), the problem of finding the largest adapted space was achieved by H. Koch and D. Tataru. They proved in [14] that the space of time-dependent divergence free vector fields on $\mathbb{R}^+ \times \mathbb{R}^3$ such that
$$\|u\|_{X_{KT}} \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac12}\|u(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} \frac{1}{R^{\frac32}}\left(\int_{P(x,R)} |u(t,y)|^2 dydt\right)^{\frac12} < \infty$$
where $P(x,R)$ is the parabolic ball $[0,R^2]\times B(x,R)$, is an adapted space.

Now let us observe that the incompressible Navier-Stokes system is translation and scaling invariant: if u is a solution of (NS) on $[0,T]\times\mathbb{R}^3$ then, for any positive λ and for any x_0 in $\mathbb{R}^3$, the vector field u_{λ,x_0} defined by
$$u_{\lambda,x_0}(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda(x-x_0))$$
is also a solution of (NS) on $[0,\lambda^{-2}T]\times\mathbb{R}^3$. Thus, an adapted space must be translation and scaling invariant in the following sense: a constant C exists such that, for any positive λ and for any x_0 in $\mathbb{R}^3$,
$$C^{-1}\|u\|_X \leq \|u_{\lambda,x_0}\|_X \leq C\|u\|_X.$$
The second term appearing in the norm $\|\cdot\|_{X_{KT}}$ above comes from the fact that the solution of (NS) should be locally in L^2 in order to be able to define the product as a locally L^1 function. The relevant norm on the initial data is $\|e^{t\Delta}u_0\|_X$. In the case of the Koch and Tataru theorem, this norm turns out to be equivalent to the norm of ∂BMO, the space of derivatives of BMO functions. Of course, the space of initial data which measures the size of the initial data must be translation and scaling invariant. A remark due to Y. Meyer (see [19]) is that the norm in such a space is always greater than the norm in the Besov space $\dot B^{-1}_{\infty,\infty}$ defined by
$$\|u\|_{\dot B^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac12}\|e^{t\Delta}u\|_{L^\infty}.$$

This leads to the following definition of a large initial data for the incompressible Navier-Stokes equations.

Definition 1.1. *A divergence free vector field u_0 is a large initial data for the incompressible Navier-Stokes system if its $\dot B^{-1}_{\infty,\infty}$ norm is large.*

Let us point out that this approach using Duhamel's formula does not use the very special structure of the incompressible Navier-Stokes system. A family of results does use the special structure of (NS): in those cases some geometrical invariance on the initial data is preserved by the flow of (NS) and this leads to some unexpected conservation of quantities, which makes the problem subcritical and thus prevents blow up. We refer for instance to [15], [18], [20], or [21], where special symmetries (like helicoidal, or axisymmetric without swirl) allow to prove global wellposedness for any data.

Some years ago, the first two authors investigated the possible existence of large initial data (in the sense of Definition 1.1) which have no preserved geometrical invariance and which nevertheless generate global regular solutions to (NS). The first result in this direction was proved in [5] where such a family of large initial data was constructed, with strong oscillations in one direction. The main point of the proof is that for any element of this family, the first iterate $\mathbb{B}(e^{t\Delta}u_0, e^{t\Delta}u_0)$ is exponentially small with respect to the large initial data u_0 in some appropriate norm. Let us notice that this result does use the fine structure of the non linear term of (NS): M. Paicu and the second author proved in [12] that for a modified incompressible Navier-Stokes system, this family of initial data generates solutions that blow up at finite time.

In [6], the first two authors constructed another class of examples, in which the initial data has slow variations in one direction. The proof of global regularity uses the fact that the 2D Navier-Stokes equations are globally wellposed. The initial data presented in the next theorem will be referred to in the following as "quasi-2D").

Theorem 1 ([6]). *Let $v_0^h = (v_0^1, v_0^2)$ be a two component, smooth divergence free vector field on $\mathbb{R}^3$ (i.e. v_0^h is in $L^2(\mathbb{R}^3)$ as well as all its derivatives), belonging, as well as all its derivatives, to $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$; let $w_0 = (w_0^h, w_0^3)$ be a three component, smooth divergence free vector field on $\mathbb{R}^3$. Then there exists a positive ε_0 such that if $\varepsilon \leq \varepsilon_0$, the initial data*

$$u_{0,\varepsilon}(x) \stackrel{\text{def}}{=} (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3)$$

generates a unique, global solution u^ε of (NS).

Remark 1.2. It is clear from the proof of [6] that the dependence of the parameter ε_0 on the profiles v_0^h and w_0 is only through their norms.

Note that such an initial data may be arbitrarily large in the sense of Definition 1.1 (see [6]). We recall for the convenience of the reader the result proved in [6].

Proposition 1.3 ([6]). *Let (f, g) be in $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R})$ and define $h_\varepsilon(x_h, x_3) \stackrel{\text{def}}{=} f(x_h)g(\varepsilon x_3)$. We have, if ε is small enough,*

$$\|h_\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

In this paper we consider the global wellposedness of the Navier-Stokes equations with data which is the sum of an initial data (which may be large) giving rise to a global solution, and a quasi-2D initial data as presented above (which may also be large). The theorem is the following.

Theorem 2. *Let u_0, v_0^h and w_0 be three smooth divergence free vector fields defined on $\mathbb{R}^3$, satisfying*

- *u_0 belongs to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and generates a unique global solution to the Navier-Stokes equations;*
- *$v_0^h = (v_0^1, v_0^2)$ is a horizontal vector field on $\mathbb{R}^3$ belonging, as well as all its derivatives, to the space $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$;*
- *$v_0^h(x_1, x_2, 0) = w_0^3(x_1, x_2, 0) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.*

Then there exists a positive number ε_0 depending on u_0 and on norms of v_0^h and w_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, there is a unique, global solution to the Navier-Stokes equations with initial data

$$u_{0,\varepsilon}(x) \stackrel{\text{def}}{=} u_0(x) + (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3).$$

Remark 1.4. Let u_0 be any element of the (open) set $\mathcal{G}$ of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ divergence free vector fields generating global smooth solutions to (NS), and let N be an arbitrarily large number. Then for any smooth divergence free vector field f^h (over $\mathbb{R}^2$) and scalar function g (over $\mathbb{R}$) satisfying $\|f^h\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})} \geq 4N$, and such that $g(0) = 0$, Theorem 2 implies that there is ε_N depending on u_0 and on norms of f^h and g such that $u_0 + (f^h \otimes g, 0)(x_1, x_2, \varepsilon_N x_3)$ belongs to $\mathcal{G}$, where we have denoted $f^h \otimes g(x) = (f^1(x_h)g(x_3), f^2(x_h)g(x_3))$. Since ε_N only depends on norms of f^h and g, that implies that for any $\lambda \in [-1, 1]$, the initial data $u_0 + \lambda(f^h \otimes g, 0)(x_1, x_2, \varepsilon_N x_3)$ also belongs to $\mathcal{G}$. Using Proposition 1.3 one concludes that: passing through u_0, there exists uncountable number of segments of length N which are included in $\mathcal{G}$.

Remark 1.5. With the notation of Theorem 2, the data $u_0(x) + (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3)$ belongs to $\mathcal{G}$ as long as ε is small enough, so one can add to that initial data any vector field of the type $(v_0^{h(1)} + \varepsilon_1 w_0^{h(1)}, w_0^{3(1)})(x_1, x_2, \varepsilon_1 x_3)$ and if ε_1 is small enough (depending on u_0, on ε, and on norms of v_0^h, w_0, $v_0^{h(1)}$ and $w_0^{(1)}$), then the resulting vector field belongs to $\mathcal{G}$. One thus immediately constructs by induction superpositions of the type

$$u_0(x) + \sum_{j=0}^{J} (v_0^{h(j)} + \varepsilon_j w_0^{h(j)}, w_0^{3(j)})(x_1, x_2, \varepsilon_j x_3)$$

which belong to $\mathcal{G}$ for small enough ε_j's, depending on u_0, on the norms of the profiles $v_0^{h(j)}$ and $w_0^{(j)}$, and on $(\varepsilon_k)_{k<j}$.

Finally notice that one can also require the slow variation on the profiles to hold on another coordinate than x_3, up to obvious modifications of the assumptions of the theorem.

Remark 1.6. In [7], an even larger initial data than the one of Theorem 1 is constructed. However the size of the solution blows up when ε tends to 0, and this is a strong obstacle to the use of a perturbative argument such as the one we will use here.

1.2. **Scheme of the proof and organization of the paper.** Let us start by introducing some notation. We shall denote by C any constant, which may change from line to line, and we will write $A \lesssim B$ if $A \leq CB$. In the following we shall denote, for any point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, its horizontal coordinates by $x_h \stackrel{\text{def}}{=} (x_1, x_2)$. Similarly the horizontal components of any vector field $u = (u^1, u^2, u^3)$ will be denoted by $u^h \stackrel{\text{def}}{=} (u^1, u^2)$ and the horizontal divergence will be defined by $\mathrm{div}_h u^h \stackrel{\text{def}}{=} \nabla^h \cdot u^h$, where $\nabla^h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$. Finally we shall define the horizontal Laplacian by $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$. We shall often use the following shorthand notation for slowly varying functions: for any function f defined on $\mathbb{R}^3$, we write

$$[f]_\varepsilon(x_h, x_3) \stackrel{\text{def}}{=} f(x_h, \varepsilon x_3). \tag{1.1}$$

In order to prove Theorem 2, we look for the solution (which exists and is smooth for a short time depending on ε, due to classical existence theory) under the form

$$u_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon^{app} + R_\varepsilon \tag{1.2}$$

where the approximate solution u_ε^{app} is defined by the sum of the global solution associated with u_0 and the quasi-2D approximation:

$$u_\varepsilon^{app} \stackrel{\text{def}}{=} u + [v_\varepsilon^{(2D)}]_\varepsilon \quad \text{with} \quad v_\varepsilon^{(2D)} \stackrel{\text{def}}{=} (v^h, 0) + (\varepsilon w_\varepsilon^h, w_\varepsilon^3) \tag{1.3}$$

while

- u is the global smooth solution of (NS) associated with the initial data u_0;

- v^h is the global smooth solution of the two dimensional Navier-Stokes equation (with parameter y_3 in $\mathbb{R}$) with pressure p_0 and data $v_0^h(\cdot, y_3)$

$$(NS2D_3) \qquad \begin{cases} \partial_t v^h + v^h \cdot \nabla^h v^h - \Delta_h v^h = -\nabla^h p_0 \\ \operatorname{div}_h v^h = 0 \\ v^h|_{t=0} = v_0^h(x_h, y_3); \end{cases}$$

- w_ε solves the linear equation with data w_0 (and pressure $p_{\varepsilon,1}$)

$$(T_v^\varepsilon) \begin{cases} \partial_t w_\varepsilon + v^h \cdot \nabla^h w_\varepsilon - \Delta_h w_\varepsilon - \varepsilon^2 \partial_3^2 w_\varepsilon = -(\nabla^h p_{\varepsilon,1}, \varepsilon^2 \partial_3 p_{\varepsilon,1}) \\ \operatorname{div} w_\varepsilon = 0 \\ w_\varepsilon|_{t=0} = w_0. \end{cases}$$

We will also define the approximate pressure

$$p_\varepsilon^{app} \stackrel{\text{def}}{=} p + [p_0 + \varepsilon p_{\varepsilon,1}]_\varepsilon. \tag{1.4}$$

The stability of this approximate solution is described by the following proposition. As in the rest of this paper, we have used the following notation: if X (resp. Y) is a function space over $\mathbb{R}^2$ (resp. $\mathbb{R}$), then we write X_h for $X(\mathbb{R}^2)$ and Y_v for $Y(\mathbb{R})$. We also denote the space $Y(\mathbb{R}; X(\mathbb{R}^2))$ by $Y_v X_h$.

Proposition 1.7. *For any positive ε_0, the family $(u_\varepsilon^{app})_{\varepsilon \le \varepsilon_0}$ of approximate solutions is uniformly bounded in $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ and the family $(\nabla u_\varepsilon^{app})_{\varepsilon \le \varepsilon_0}$ is uniformly bounded in $L^2(\mathbb{R}^+; L_v^\infty(L_h^2))$.*

The size of the error term E_ε (this denomination will become apparent in the next section) defined by

$$E_\varepsilon \stackrel{\text{def}}{=} (\partial_t - \Delta) u_\varepsilon^{app} + u_\varepsilon^{app} \cdot \nabla u_\varepsilon^{app} + \nabla p_\varepsilon^{app} \tag{1.5}$$

can be estimated as follows.

Proposition 1.8. *The family $(E_\varepsilon)_{\varepsilon \le \varepsilon_0}$ of error terms satisfies*

$$\lim_{\varepsilon \to 0} \|E_\varepsilon\|_{L^2(\mathbb{R}^+; \dot{H}^{-\frac{1}{2}})} = 0.$$

The structure of this article is the following:

- the second section is devoted the proof of Theorem 2 using the above two propositions;
- the third section consists in proving Proposition 1.8 using estimates on the product in anisotropic spaces;
- we shall present the proof of some product laws in Sobolev spaces in Appendix A;
- the proof of Proposition 1.7 is postponed to Appendix B. Indeed most of the proof is actually contained in Lemma 2.1 of [6], apart from the fact that the global solution u satisfies the required properties. One way to avoid having to rely on that last result would be simply to replace, in the definition of u_ε^{app}, the solution u by any smooth approximation (that is possible due to the stability result of [11]). However we feel the result in itself is interesting so we prove in Appendix B that any global solution associated to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ initial data belongs to $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$, and its gradient to $L^2(\mathbb{R}^+; L_v^\infty(L_h^2))$.

2. Proof of Theorem 2

Assuming Proposition 1.8, the proof of Theorem 2 follows the sames lines as the proof of Theorem 3 of [6]; we recall it for the reader's convenience. Using the definition of the approximate solution $(u_\varepsilon^{app}, p_\varepsilon^{app})$ given in (1.3,1.4), and the error term E_ε given in (1.5), we find that the remainder R_ε satisfies the following modified three-dimensional Navier-Stokes equation

$$
(MNS_\varepsilon) \qquad \left\{ \begin{array}{c} \partial_t R_\varepsilon + R_\varepsilon \cdot \nabla R_\varepsilon - \Delta R_\varepsilon + u_\varepsilon^{app} \cdot \nabla R_\varepsilon + R_\varepsilon \cdot \nabla u_\varepsilon^{app} = -E_\varepsilon - \nabla q_\varepsilon \\ \operatorname{div} R_\varepsilon = 0 \quad \text{and} \quad R_\varepsilon|_{t=0} = 0, \end{array} \right.
$$

with $q_\varepsilon \stackrel{\text{def}}{=} p_\varepsilon - p_\varepsilon^{app}$. The proof of the theorem reduces to the proof that (MNS_ε) is globally wellposed. We shall only write the useful a priori estimates on R_ε, and leave to the reader the classical arguments allowing to deduce the result. In particular we omit the proof of the fact that the solution R_ε constructed in this way is continuous in time with values in $\dot H^{\frac12}(\mathbb{R}^3)$.

So let us define, for any $\lambda > 0$,

$$
R_\varepsilon^\lambda(t) \stackrel{\text{def}}{=} R_\varepsilon(t) \exp\left(-\lambda \int_0^t V_\varepsilon(t')\,dt'\right) \quad \text{with} \quad V_\varepsilon(t) \stackrel{\text{def}}{=} \|u_\varepsilon^{app}(t)\|_{L^\infty}^2 + \|\nabla u_\varepsilon^{app}(t)\|_{L^\infty_v L^2_h}^2.
$$

Note that Proposition 1.7 implies that $\displaystyle\int_{\mathbb{R}^+} V_\varepsilon(t)\,dt$ is uniformly bounded, by a constant denoted by U in the following. Writing also $E_\varepsilon^\lambda(t) \stackrel{\text{def}}{=} E_\varepsilon(t) \exp\left(-\lambda \displaystyle\int_0^t V_\varepsilon(t')\,dt'\right)$, an $\dot H^{\frac12}$ energy estimate on (MNS_ε) implies

$$
\begin{aligned}
\frac12 \frac{d}{dt} \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 &= -\lambda V_\varepsilon(t) \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 - \big(E_\varepsilon^\lambda | R_\varepsilon^\lambda\big)_{\dot H^{\frac12}}(t) \\
&\quad - \Big(\exp\Big(\lambda \int_0^t V_\varepsilon(t')dt'\Big) R_\varepsilon^\lambda \cdot \nabla R_\varepsilon^\lambda + u_\varepsilon^{app} \cdot \nabla R_\varepsilon^\lambda + R_\varepsilon^\lambda \cdot \nabla u_\varepsilon^{app} \big| R_\varepsilon^\lambda\Big)_{\dot H^{\frac12}}(t).
\end{aligned}
$$

A law of product in Sobolev spaces (see (A.2) in Appendix A) and Proposition 1.7 imply that

$$
\begin{aligned}
\exp\Big(\lambda \int_0^t V_\varepsilon(t')dt'\Big) \big|(R_\varepsilon^\lambda \cdot \nabla R_\varepsilon^\lambda | R_\varepsilon^\lambda)_{\dot H^{\frac12}}\big| &\le C e^{\lambda U} \|R_\varepsilon^\lambda(t)\|_{\dot H^1}^2 \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}} \\
&\le C e^{\lambda U} \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}} \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2.
\end{aligned}
$$

Lemma 2.3 of [6] claims that

$$
\big|(b \cdot \nabla a + a \cdot \nabla b | b)_{\dot H^{\frac12}}\big| \le C\big(\|a\|_{L^\infty} + \|\nabla a\|_{L^\infty_v L^2_h}\big) \|b\|_{\dot H^{\frac12}} \|\nabla b\|_{\dot H^{\frac12}}, \tag{2.1}
$$

so by definition of V_ε we get

$$
\big|(u_\varepsilon^{app} \cdot \nabla R_\varepsilon^\lambda + R_\varepsilon^\lambda \cdot \nabla u_\varepsilon^{app} | R_\varepsilon^\lambda)_{\dot H^{\frac12}}\big| \le \frac14 \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + C V_\varepsilon(t) \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2.
$$

Let us choose $\lambda \ge C$. Then using the fact that

$$
\big|(E_\varepsilon^\lambda | R_\varepsilon^\lambda)_{\dot H^{\frac12}}(t)\big| \le \frac14 \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + C \|E_\varepsilon^\lambda(t)\|_{\dot H^{-\frac12}}^2
$$

we obtain

$$
\frac{d}{dt} \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + \frac32 \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 \le C \|E_\varepsilon(t)\|_{\dot H^{-\frac12}}^2 + C e^{CU} \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}} \|\nabla R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2.
$$

Since $R_{\varepsilon|t=0} = 0$ and $\lim\limits_{\varepsilon \to 0} \|E_\varepsilon\|_{L^2(\mathbb{R}^+; \dot H^{-\frac12})} = 0$ by Proposition 1.8, we deduce that as long as $\|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}$ is smaller than $(4Ce^{CU})^{-1}$, then for any $\eta > 0$ there is ε_0 such that

$$
\forall \varepsilon \le \varepsilon_0, \quad \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + \frac34 \int_0^t \|\nabla R_\varepsilon^\lambda(t')\|_{\dot H^{\frac12}}^2\,dt' \le \eta,
$$

which in turn implies that

$$
\forall \varepsilon \le \varepsilon_0, \quad \forall t \in \mathbb{R}^+, \quad \|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}^2 + \frac34 \int_0^t \|\nabla R_\varepsilon^\lambda(t')\|_{\dot H^{\frac12}}^2\,dt' \le \eta.
$$

That concludes the proof of the theorem. $\square$

3. The estimate of the error term

In this section, we shall prove Proposition 1.8. Let us first remark that the error term E_ε can be decomposed as

$$E_\varepsilon = E^1_\varepsilon + E^2_\varepsilon \quad \text{with} \quad E^2_\varepsilon \stackrel{\text{def}}{=} u\cdot\nabla[v^{(2D)}_\varepsilon]_\varepsilon + [v^{(2D)}_\varepsilon]_\varepsilon\cdot\nabla u.$$

Thus the term E^1_ε is exactly the error term which appears in [6], and Lemmas 4.1, 4.2 and 4.3 of [6] imply that

$$\|E^1_\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12})} \leq C_0\varepsilon^{\frac13}. \tag{3.2}$$

In order to estimate the term E^2_ε, let us first observe that Lemmas 3.1 and 3.2 of [6] imply the following proposition.

Proposition 3.1 ([6])**.** *For any s greater than -1, for any $\alpha\in\mathbb{N}^3$ and for any positive t, we have*

$$\|\partial^\alpha v^{(2D)}_\varepsilon(t)\|^2_{L^2_v\dot H^s_h} + \int_0^t \|\partial^\alpha\nabla^h v^{(2D)}_\varepsilon(t')\|^2_{L^2_v\dot H^s_h}dt' \leq C_0.$$

We shall also be using the following result, whose proof is postponed to the end of this paragraph.

Proposition 3.2. *The vector field $v^{(2D)}_\varepsilon$ satisfies*

$$\|v^{(2D)}_\varepsilon(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)} + \|\nabla^h v^{(2D)}_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)} \leq C\varepsilon^{\frac12}. \tag{3.3}$$

Furthermore, $v^{(2D)}_\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}^+,\dot H^{\frac12}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+,\dot H^{\frac32}(\mathbb{R}^3))$.

Assuming this result, let us prove Proposition 1.8.

Proof of Proposition 1.8: The stability theorem of [11] claims in particular that

$$\lim_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}} = 0.$$

As the set of smooth compactly supported divergence free vector fields is dense in the space of $\dot H^{\frac12}(\mathbb{R}^3)$ divergence free vector fields, this allows to construct for any positive η, a family $(t_j)_{1\leq j\leq N}$ of positive real numbers and a family $(\phi_j)_{1\leq j\leq N}$ of smooth compactly supported divergence free vector fields such that (with $t_0=0$)

$$\|\underline{u}_\eta\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})} \leq \eta \quad\text{with}\quad \underline{u}_\eta \stackrel{\text{def}}{=} u-\widetilde u_\eta \quad\text{and}\quad \widetilde u_\eta(t,x) \stackrel{\text{def}}{=} \sum_{j=1}^N \mathbf{1}_{[t_{j-1},t_j]}(t)\phi_j(x). \tag{3.4}$$

Then, for any positive η, let us decompose E^2_ε as

$$E^2_\varepsilon = \underline{E}_{\varepsilon,\eta} + \widetilde E_{\varepsilon,\eta} \quad\text{with}\quad \underline{E}_{\varepsilon,\eta} \stackrel{\text{def}}{=} \underline{u}_\eta\cdot\nabla[v^{(2D)}_\varepsilon]_\varepsilon + [v^{(2D)}_\varepsilon]_\varepsilon\cdot\nabla\underline{u}_\eta. \tag{3.5}$$

The term $\underline{E}_{\varepsilon,\eta}$ will be estimated thanks to the following lemma which is a generalization of (2.1).

Lemma 3.3. *Let a and b be two smooth functions. We have*

$$\|ab\|_{\dot H^{\frac12}} \leq C\|a\|_{\dot H^{\frac12}}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)} + \|b\|_{L^\infty} + \|\partial_3 b\|_{L^2_v(\dot H^{\frac12}_h)}\big).$$

Proof. For any function f in $\dot H^{\frac12}(\mathbb{R}^3)$, one has

$$\|f\|_{\dot H^{\frac12}} \leq \|f\|_{L^2_h \dot H^{\frac12}_v} + \|f\|_{L^2_v \dot H^{\frac12}_h}. \tag{3.6}$$

That estimate may be proved simply by Plancherel's theorem (see for instance the end of the proof of Lemma 2.3 of [6]).

Now we observe that by two-dimensional product laws (taking $s=\frac12$ and $d=2$ in (A.1) of Appendix A), one has for any x_3 in $\mathbb{R}$

$$\|a(\cdot,x_3)b(\cdot,x_3)\|_{\dot H^{\frac12}_h} \leq C\big(\|a(\cdot,x_3)\|_{\dot H^{\frac12}_h}\|\nabla^h b(\cdot,x_3)\|_{L^2_h} + \|a(\cdot,x_3)\|_{\dot H^{\frac12}_h}\|b(\cdot,x_3)\|_{L^\infty_h}\big).$$

One has of course

$$s\leq 0 \Longrightarrow \|a\|_{\dot H^s} \leq \|a\|_{L^2_v(\dot H^s_h)} \quad\text{and}\quad s\geq 0 \Longrightarrow \|a\|_{L^2_v(\dot H^s_h)} \leq \|a\|_{\dot H^s} \tag{3.7}$$

so taking $s=1/2$ gives

$$\begin{aligned}\|ab\|_{L^2_v\dot H^{\frac12}_h} &\leq C\|a\|_{L^2_v(\dot H^{\frac12}_h)}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)} + \|b\|_{L^\infty}\big)\\ &\leq \|a\|_{\dot H^{\frac12}}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)} + \|b\|_{L^\infty}\big).\end{aligned} \tag{3.8}$$

Now let us estimate $\|ab\|_{L^2_h\dot H^{\frac12}_v}$. A law of product in the vertical variable (taking $s=\frac12$ and $d=1$ in (A.1) of Appendix A) implies that for any x_h in $\mathbb{R}^2$

$$\|a(x_h,\cdot)b(x_h,\cdot)\|_{\dot H^{\frac12}_v} \leq C\big(\|a(x_h,\cdot)\|_{\dot H^{\frac12}_v}\|b(x_h,\cdot)\|_{L^\infty_v} + \|a(x_h,\cdot)\|_{L^2_v}\|\partial_3 b(x_h,\cdot)\|_{L^2_v}\big).$$

Taking the L^2 norm in the horizontal variable gives

$$\|ab\|_{L^2_h\dot H^{\frac12}_v} \leq C\big(\|a\|_{L^2_h\dot H^{\frac12}_v}\|b\|_{L^\infty} + \|a\|_{L^4_hL^2_v}\|\partial_3 b\|_{L^4_hL^2_v}\big).$$

Using Minkowski's inequality, we get that

$$\|ab\|_{L^2_h\dot H^{\frac12}_v} \leq C\big(\|a\|_{L^2_h\dot H^{\frac12}_v}\|b\|_{L^\infty} + \|a\|_{L^2_vL^4_h}\|\partial_3 b\|_{L^2_vL^4_h}\big).$$

Then using the Sobolev embedding $\dot H^{\frac12}_h \hookrightarrow L^4_h$ and (3.7), we infer

$$\begin{aligned}\|ab\|_{L^2_h\dot H^{\frac12}_v} &\leq C\big(\|a\|_{L^2_h\dot H^{\frac12}_v}\|b\|_{L^\infty} + \|a\|_{L^2_v\dot H^{\frac12}_h}\|\partial_3 b\|_{L^2_v\dot H^{\frac12}_h}\big)\\ &\leq C\|a\|_{\dot H^{\frac12}}\big(\|b\|_{L^\infty} + \|\partial_3 b\|_{L^2_v\dot H^{\frac12}_h}\big).\end{aligned}$$

Together with (3.6) and (3.8), this proves Lemma 3.3. □

That lemma allows to obtain the required estimate for $\underline{E}_{\varepsilon,\eta}$. Using the divergence free condition, we indeed have that

$$\underline{E}_{\varepsilon,\eta} = \operatorname{div}(\underline{u}_\eta \otimes [v^{(2D)}_\varepsilon]_\varepsilon + [v^{(2D)}_\varepsilon]_\varepsilon \otimes \underline{u}_\eta).$$

So the above lemma implies that for any positive time t

$$\begin{aligned}\|\underline{E}_{\varepsilon,\eta}(t)\|_{\dot H^{-\frac12}} &\leq C\|\underline{u}_\eta \otimes [v^{(2D)}_\varepsilon]_\varepsilon + [v^{(2D)}_\varepsilon]_\varepsilon \otimes \underline{u}_\eta\|_{\dot H^{\frac12}}(t)\\ &\leq C\|\underline{u}_\eta\|_{\dot H^{\frac12}}\big(\|\nabla^h[v^{(2D)}_\varepsilon]_\varepsilon\|_{L^\infty_vL^2_h} + \|[v^{(2D)}_\varepsilon]_\varepsilon\|_{L^\infty} + \|\partial_3[v^{(2D)}_\varepsilon]_\varepsilon\|_{L^2_v\dot H^{\frac12}_h}\big)(t).\end{aligned}$$

By definition of $[\,\cdot\,]_\varepsilon$ and using (3.4), we get

$$\|\underline{E}_{\varepsilon,\eta}(t)\|_{\dot H^{-\frac12}} \leq C\eta\big(\|\nabla^h v^{(2D)}_\varepsilon\|_{L^\infty_vL^2_h} + \|v^{(2D)}_\varepsilon\|_{L^\infty} + \varepsilon^{\frac12}\|\partial_3 v^{(2D)}_\varepsilon\|_{L^2_v\dot H^{\frac12}_h}\big).$$

Proposition 3.1, along with Proposition 1.7, gives finally

$$\|\underline{E}_{\varepsilon,\eta}\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12})} \leq C_0\eta. \tag{3.9}$$

In order to estimate the term $\widetilde{E}_{\varepsilon,\eta}$ let us observe that thanks to the divergence free condition, we have

$$\widetilde{E}_{\varepsilon,\eta} = \widetilde{u}^h_\eta \cdot \nabla^h [v_\varepsilon^{(2D)}]_\varepsilon + \varepsilon \widetilde{u}^3_\eta [\partial_3 v_\varepsilon^{(2D)}]_\varepsilon + [v_\varepsilon^{(2D)}]_\varepsilon \cdot \nabla \widetilde{u}_\eta. \tag{3.10}$$

Using a 3D law of product (namely (A.2) in Appendix A) gives

$$\|\varepsilon \widetilde{u}^3_\eta [\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot H^{-\frac12}} \leq C\varepsilon \|\widetilde{u}_\eta\|_{\dot H^{\frac12}} \|[\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot H^{\frac12}}.$$

This gives

$$\|\varepsilon \widetilde{u}^3_\eta [\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot H^{-\frac12}} \leq \varepsilon^{\frac12} \|\widetilde{u}_\eta\|_{\dot H^{\frac12}} \|v_\varepsilon^{(2D)}\|_{\dot H^{\frac32}}. \tag{3.11}$$

The two other terms of (3.10) are estimated using the following lemma.

Lemma 3.4. *Let a and b be two smooth functions. We have*

$$\|ab\|_{\dot H^{-\frac12}} \leq C\|a\|_{L^2_v \dot H^{\frac12}_h} \|b(\cdot,0)\|_{L^2_h} + C\|x_3 a\|_{L^2} \|\partial_3 b\|_{L^\infty_v \dot H^{\frac12}_h}.$$

Proof. Let us decompose b in the following way:

$$b(x_h,x_3) = b(x_h,0) + \int_0^{x_3} \partial_3 b(x_h,y_3)\,dy_3. \tag{3.12}$$

Laws of product for Sobolev spaces on $\mathbb{R}^2$ (see (A.2) in Appendix A) together with Assertion (3.7) gives

$$\begin{aligned}
\|a(b_{|x_3=0})\|_{\dot H^{-\frac12}} &\leq \|a(b_{|x_3=0})\|_{L^2_v \dot H^{-\frac12}_h} \\
&\leq \left(\int_{\mathbb{R}} \|a(\cdot,x_3) b(\cdot,0)\|^2_{\dot H^{-\frac12}_h} dx_3\right)^{\frac12} \\
&\leq C\|b(\cdot,0)\|_{L^2}\left(\int_{\mathbb{R}} \|a(\cdot,x_3)\|^2_{\dot H^{\frac12}_h} dx_3\right)^{\frac12} \\
&\leq C\|a\|_{L^2_v \dot H^{\frac12}_h} \|b(\cdot,0)\|_{L^2_h}. \qquad (3.13)
\end{aligned}$$

In order to use (3.12), let us observe that for any x_3, two-dimensional product laws give

$$\begin{aligned}
\left\| a(\cdot,x_3)\int_0^{x_3} \partial_3 b(\cdot,y_3)dy_3\right\|_{\dot H^{-\frac12}_h} &\leq C\|a(\cdot,x_3)\|_{L^2_h} \left|\int_0^{x_3} \|\partial_3 b(\cdot,y_3)\|_{\dot H^{\frac12}_h} dy_3\right| \\
&\leq C|x_3|\|a(\cdot,x_3)\|_{L^2_h} \|\partial_3 b\|_{L^\infty_v \dot H^{\frac12}_h}.
\end{aligned}$$

The above estimate integrated in x_3 together with (3.12) and (3.13) gives the result. □

Now let us apply this lemma to estimate $\widetilde{u}^h_\eta \cdot \nabla^h [v_\varepsilon^{(2D)}]_\varepsilon$ and $[v_\varepsilon^{(2D)}]_\varepsilon \cdot \nabla \widetilde{u}_\eta$. We get

$$\begin{aligned}
\|\widetilde{u}^h_\eta \cdot \nabla^h [v_\varepsilon^{(2D)}]_\varepsilon(t)\|_{\dot H^{-\frac12}} \leq C\|\widetilde{u}^h_\eta(t,\cdot)\|_{L^2_v(\dot H^{\frac12}_h)} &\|\nabla^h v_\varepsilon^{(2D)}(t,\cdot,0)\|_{L^2_h} \\
&+ \varepsilon \|x_3 \widetilde{u}^h_\eta(t)\|_{L^2} \|\partial_3 \nabla^h v_\varepsilon^{(2D)}(t,\cdot)\|_{L^\infty_v(\dot H^{\frac12}_h)}
\end{aligned}$$

and

$$\begin{aligned}
\|[v_\varepsilon^{(2D)}]_\varepsilon \cdot \nabla \widetilde{u}_\eta(t)\|_{\dot H^{-\frac12}} \leq C\|\nabla \widetilde{u}_\eta(t,\cdot)\|_{L^2_v(\dot H^{\frac12}_h)} &\|v_\varepsilon^{(2D)}(t,\cdot,0)\|_{L^2_h} \\
&+ \varepsilon \|x_3 \nabla \widetilde{u}_\eta(t,\cdot)\|_{L^2} \|\partial_3 v_\varepsilon^{(2D)}(t,\cdot)\|_{L^\infty_v(\dot H^{\frac12}_h)}.
\end{aligned}$$

By construction of $\widetilde{u}_\eta$ and by Proposition 3.1 and 3.2 (using the embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$), together with (3.10) and (3.11), we infer that

$$\|\widetilde{E}_{\varepsilon,\eta}\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12})} \leq C_\eta \varepsilon^{\frac12} \tag{3.14}$$

and putting (3.2), (3.9) and (3.14) together proves Proposition 1.8, up to the proof of Proposition 3.2. $\square$

Let us finally prove Proposition 3.2.

Proof of Proposition 3.2. We recall that $v_\varepsilon^{(2D)} = (v^h,0) + (\varepsilon w_\varepsilon^h, w_\varepsilon^3)$, and due to the form of $(NS2D_3)$ it is clear that $v^h(t,x_h,0) = 0$ for any (t,x_h) in $\mathbb{R}^+ \times \mathbb{R}^2$. So it remains to estimate $(\varepsilon w_\varepsilon^h, w_\varepsilon^3)$. We first notice that due to Lemma 3.2 of [6],

$$\|\varepsilon w_\varepsilon^h(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)} + \|\varepsilon \nabla^h w_\varepsilon^h(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)} \leq C\varepsilon,$$

so we are left with the computation of $w_\varepsilon^3(t,\cdot,0)$. By definition of w_ε we have

$$\begin{cases} \partial_t w_\varepsilon^3 + v^h \cdot \nabla^h w_\varepsilon^3 - \Delta_h w_\varepsilon^3 = \varepsilon^2 F_\varepsilon \\ w^3_{\varepsilon|t=0} = w_0^3 \end{cases} \quad \text{with} \quad F_\varepsilon \stackrel{\text{def}}{=} \partial_3^2 w_\varepsilon^3 - \partial_3 p_{\varepsilon,1}.$$

We shall start by writing an $\dot{H}_h^{\frac12}$ energy estimate (with y_3 seen as a parameter) which will imply that $w_\varepsilon^3(t,\cdot,0)$ is smaller than $C\varepsilon$ in $L^\infty(\mathbb{R}^+;\dot{H}_h^{\frac12}) \cap L^2(\mathbb{R}^+;\dot{H}_h^{\frac32})$. The result in the space $L^\infty(\mathbb{R}^+;L^2_h) \cap L^2(\mathbb{R}^+;\dot{H}^1_h)$ will follow by interpolation with a bound in a negative order Sobolev space, given by Lemma 3.2 of [6].

Let us start by the $\dot{H}_h^{\frac12}$ energy estimate. We claim that there is a constant C_0 such that for any $\varepsilon \leq \varepsilon_0$,

$$\varepsilon \|F_\varepsilon\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})} \leq C_0. \tag{3.15}$$

Assuming (3.15), an $\dot{H}_h^{\frac12}$ energy estimate (joint with the fact that $w^3_{\varepsilon|t=0}(\cdot,0) = 0$) gives directly that

$$\|w_\varepsilon^3(\cdot,0)\|_{L^\infty(\mathbb{R}^+;\dot{H}_h^{\frac12})} + \|\nabla^h w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+;\dot{H}_h^{\frac12})} \leq C\varepsilon.$$

But by Lemma 3.2 of [6] we know that w_ε^3 is uniformly bounded, say in $L^\infty(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})$ and $\nabla^h w_\varepsilon^3$ is uniformly bounded in $L^2(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})$, so we get by interpolation that

$$\|w_\varepsilon^3(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)} + \|\nabla^h w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)} \leq C\varepsilon^{\frac12}.$$

This achieves (3.3).

It remains to prove the claim (3.15). On the one hand, Lemma 3.2 of [6] implies that

$$\|\partial_3^2 w_\varepsilon^3\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})} = \|\partial_3 \nabla^h \cdot w_\varepsilon^h\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})} \leq C_0. \tag{3.16}$$

The estimate on the pressure seems slightly more delicate, but we notice as in [6] that

$$-(\varepsilon^2 \partial_3^2 + \Delta_h) p_{\varepsilon,1} = \operatorname{div}_h\big(v^h \cdot \nabla^h w_\varepsilon^h + \partial_3(w_\varepsilon^3 v^h)\big). \tag{3.17}$$

Since $\varepsilon\partial_3 \operatorname{div}_h(\varepsilon^2\partial_3^2 + \Delta_h)^{-1}$ is a uniformly bounded Fourier multiplier, this implies by Sobolev embedding that

$$\begin{aligned} \|\varepsilon \partial_3 p_{\varepsilon,1}\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}_h^{-\frac12})} &\lesssim \|\varepsilon \partial_3 p_{\varepsilon,1}\|_{L^2(\mathbb{R}^+;H^1_v \dot{H}_h^{-\frac12})} \\ &\lesssim \|v^h \cdot \nabla^h w_\varepsilon^h + \partial_3 w_\varepsilon^3 v^h + w_\varepsilon^3 \partial_3 v^h\|_{L^2(\mathbb{R}^+;H^1_v \dot{H}_h^{-\frac12})}. \end{aligned}$$

However thanks to (A.2) and using the estimates of Lemmas 3.1 and 3.2 of [6], one has

$$\|v^h\cdot\nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+;H^1_v\dot H_h^{-\frac12})}\leq C\|v^h\|_{L^\infty(\mathbb{R}^+;H^1_v\dot H_h^{\frac12})}\|\nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+;H^1_vL^2_h)}\leq C.$$

Due to the divergence free condition of w_ε, a similar estimate holds for $\|\partial_3 w_\varepsilon^3 v^h\|_{L^2(\mathbb{R}^+;H^1_v\dot H_h^{-\frac12})}$. While again thanks to (A.2) and using the estimates of Lemmas 3.1 and 3.2 of [6], we obtain

$$\|w_\varepsilon^3\partial_3 v^h\|_{L^2(\mathbb{R}^+;H^1_v\dot H_h^{-\frac12})}\leq C\|w_\varepsilon^3\|_{L^2(\mathbb{R}^+;H^1_v\dot H_h^{\frac12})}\|\partial_3 v^h\|_{L^\infty(\mathbb{R}^+;H^1_vL^2_h)}\leq C.$$

As a consequence, we arrive at

$$\|\varepsilon\partial_3 p_{\varepsilon,1}\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H_h^{-\frac12})}\leq C_0. \tag{3.18}$$

The combination of (3.16) and (3.18) proves the claim, hence Estimate (3.3) of Proposition 3.2.

Finally let us prove the bound in $L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))\cap L^2(\mathbb{R}^+,\dot H^{\frac32}(\mathbb{R}^3))$ of $v_\varepsilon^{(2D)}$. Actually the bound for $(v^h,0)$ follows from Lemma 3.1 and Corollary 3.1 of [6], so we just have to concentrate on $(\varepsilon w_\varepsilon^h, w_\varepsilon^3)$. Lemma 3.2 of [6] gives that w_ε is uniformly bounded in $L^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))$, as well as the fact that $\nabla^h w_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))$ so by the divergence free condition we only need to check that $\varepsilon\partial_3 w_\varepsilon^h$ is uniformly bounded in $L^2(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))$. In fact, we shall prove first that $\varepsilon\partial_3 w_\varepsilon^h$ is uniformly bounded in $L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))$ and then that $(\varepsilon\partial_3)^2 w_\varepsilon^h$ is uniformly bounded in $L^2(\mathbb{R}^+;L^2(\mathbb{R}^3))$, so that the result will follow by interpolation, using Lemma 3.2 of [6] to deal with horizontal derivatives. Actually we shall only concentrate on the first bound and leave the second to the reader as it is very similar. Indeed we get by a standard L^2 energy estimate on w_ε^h that

$$\frac12\frac{d}{dt}\|w_\varepsilon^h\|_{L^2}^2+\|\nabla^h w_\varepsilon^h\|_{L^2}^2+\|\varepsilon\partial_3 w_\varepsilon^h\|_{L^2}^2=-(v^h\cdot\nabla^h w_\varepsilon^h+\nabla^h p_{\varepsilon,1}|w_\varepsilon^h)_{L^2}.$$

On the one hand we can write

$$|(v^h\cdot\nabla^h w_\varepsilon^h|w_\varepsilon^h)_{L^2}|\leq C\|v^h\|_{L^\infty}\|\nabla^h w_\varepsilon^h\|_{L^2}\|w_\varepsilon^h\|_{L^2}$$

which implies that

$$|(v^h\cdot\nabla^h w_\varepsilon^h|w_\varepsilon^h)_{L^2}|\leq\frac14\|\nabla^h w_\varepsilon^h\|_{L^2}^2+C\|w_\varepsilon^h\|_{L^2}^2\|v^h\|_{L^\infty}^2.$$

To estimate the pressure term, we use again (3.17) which allows to write (using the fact that $\partial_3 w_\varepsilon^3=-\operatorname{div}_h w_\varepsilon^h$)

$$\begin{aligned}|(\nabla^h p_{\varepsilon,1}|w_\varepsilon^h)_{L^2}|&\leq C\int_{\mathbb{R}}\|w_\varepsilon^h\|_{\dot H_h^{\frac12}}\Big(\|v^h\|_{\dot H_h^{\frac12}}\|\nabla^h w_\varepsilon^h\|_{L^2_h}+\|w_\varepsilon^3\|_{\dot H_h^{\frac12}}\|\partial_3 v^h\|_{L^2_h}\Big)\,dx_3\\&\leq C\|w_\varepsilon^h\|_{L^2_v\dot H_h^{\frac12}}\Big(\|v^h\|_{L^\infty_v\dot H_h^{\frac12}}\|\nabla^h w_\varepsilon^h\|_{L^2}+\|w_\varepsilon^3\|_{L^2_v\dot H_h^{\frac12}}\|\partial_3 v^h\|_{L^\infty_vL^2_h}\Big).\end{aligned}$$

This implies, after some interpolation estimates, that

$$|(\nabla^h p_{\varepsilon,1}|w_\varepsilon^h)_{L^2}|\leq\frac14\|\nabla^h w_\varepsilon^h\|_{L^2}^2+C\|w_\varepsilon^h\|_{L^2}^2\|v^h\|^4_{L^\infty_v\dot H_h^{\frac12}}+C\|w_\varepsilon\|^2_{L^2_v\dot H_h^{\frac12}}\|\partial_3 v^h\|_{L^\infty_vL^2_h}.$$

Thus applying Gronwall's lemma ensures that

$$\begin{aligned}\|w_\varepsilon^h(t)\|_{L^2}^2 &+ \int_0^t \|\nabla^h w_\varepsilon^h(t')\|_{L^2}^2\,dt' + \int_0^t \|\varepsilon\partial_3 w_\varepsilon^h(t')\|_{L^2}^2\,dt' \\ &\leq \big(C\|w_\varepsilon\|^2_{L^2(\mathbb{R}^+;L^2_v\dot H_h^{\frac12})}\|\partial_3 v^h\|_{L^\infty(\mathbb{R}^+;L^\infty_v L^2_h)} + \|w_\varepsilon^h(0)\|_{L^2}^2\big) \\ &\qquad\times \exp C\Big(\|v^h\|^4_{L^4(\mathbb{R}^+;L^\infty_v\dot H_h^{\frac12})} + \|v^h\|^2_{L^2(\mathbb{R}^+;L^\infty)}\Big),\end{aligned}$$

so the results of Lemmas 3.1, 3.2 and Corollary 3.1 of [6] allow to conclude that $(\varepsilon\partial_3)w_\varepsilon^h$ is uniformly bounded in $L^2(\mathbb{R}^+,L^2(\mathbb{R}^3))$. The estimates are similar for $(\varepsilon\partial_3)^2 w_\varepsilon^h$, and that concludes the proof of the proposition. □

Appendix A. Product laws in $\dot H^s(\mathbb{R}^d)$

To prove the product laws in $\dot H^s(\mathbb{R}^d)$ as well as Proposition B.1 below, we shall need some basic facts on Littlewood-Paley analysis, which we shall recall here without proof but refer for instance to [1] for all necessary details. Let $\widehat\phi$ (the Fourier transform of ϕ) be a radial function in $\mathcal{D}(\mathbb{R}^d)$ such that $\widehat\phi(\xi)=1$ for $|\xi|\leq 1$ and $\widehat\phi(\xi)=0$ for $|\xi|>2$, and we define $\phi_\ell(x)=2^{d\ell}\phi(2^\ell x)$. Then the frequency localization operators are defined by

$$S_\ell = \phi_\ell * \cdot \quad \text{and} \quad \Delta_\ell = S_{\ell+1} - S_\ell.$$

Let f be in $\mathcal{S}'(\mathbb{R}^d)$, let p,q belong to $[1,\infty]$, and let $s<d/p$. We say that f belongs to $\dot B^s_{p,q}(\mathbb{R}^d)$ if and only if

- The partial sum $\sum_{-m}^m \Delta_\ell f$ converges to f as a tempered distribution;
- The sequence $\varepsilon_\ell = 2^{\ell s}\|\Delta_\ell f\|_{L^p}$ belongs to ℓ^q.

We will also need a slight modification of those spaces, taking into account the time variable; we refer to [8] for the introduction of that type of space in the context of the Navier–Stokes equations. Let $u(t,x)\in\mathcal{S}'(\mathbb{R}^{1+d})$ and let Δ_ℓ be a frequency localization with respect to the x variable. We will say that $u\in\widetilde{L^\rho}(\mathbb{R}^+;\dot B^s_{p,q}(\mathbb{R}^d))$ if and only if

$$2^{\ell s}\|\Delta_\ell u\|_{L^\rho(\mathbb{R}^+;L^p)} = \varepsilon_\ell \in \ell^q,$$

and other requirements are the same as in the previous definition. Note that there is an equivalent definition of Besov spaces in terms of the heat flow: for any positive s,

$$\|u\|_{\dot B^{-s}_{p,r}} = \Big\|t^{\frac{s}{2}}\|e^{t\Delta}u(t)\|_{L^p}\Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})}.$$

Now let us apply the above facts to study product laws in $\dot H^s(\mathbb{R}^d)$. The proofs are very classical (see [1] for instance), and we present them here just for the readers' convenience.

Proposition A.1. *(i) Let $a\in\dot H^{\frac{d-1}{2}}(\mathbb{R}^d)\cap\dot H^s(\mathbb{R}^d)$ and $b\in L^\infty(\mathbb{R}^d)\cap\dot H^{s+\frac12}(\mathbb{R}^d)$ for $s>0$. Then $ab\in\dot H^s(\mathbb{R}^d)$ and*

$$\|ab\|_{\dot H^s} \lesssim \|a\|_{\dot H^{\frac{d-1}{2}}}\|b\|_{\dot H^{s+\frac12}} + \|a\|_{\dot H^s}\|b\|_{L^\infty}. \tag{A.1}$$

(ii) Let $a\in\dot H^{s_1}(\mathbb{R}^d)$, $b\in\dot H^{s_2}(\mathbb{R}^d)$ with $s_1+s_2>0$ and $s_1,s_2<\frac d2$. Then $ab\in\dot B^{s_1+s_2-\frac d2}_{2,1}(\mathbb{R}^d)$, and

$$\|ab\|_{\dot B^{s_1+s_2-\frac d2}_{2,1}} \lesssim \|a\|_{\dot H^{s_1}}\|b\|_{\dot H^{s_2}}. \tag{A.2}$$

Proof. In what follows $(c_j)_{j\in\mathbb{Z}}$ (resp. $(d_j)_{j\in\mathbb{Z}}$) will always be a generic element in the sphere of ℓ^2 (resp. ℓ^1).

Thanks to Bony's decomposition [2], we have $ab = T_a b + T_b a + R(a,b)$, with

$$T_a b = \sum_{j\in\mathbb{Z}} S_{j-1}a\Delta_j b \quad\text{and}\quad R(a,b) = \sum_{j\in\mathbb{Z}} \Delta_j a\widetilde{\Delta}_j b, \quad\text{while}\quad \widetilde{\Delta}_j b = \sum_{\ell=-1}^{1} \Delta_{j+\ell} b.$$

(i) Bernstein's inequalities give

$$\|S_j a\|_{L^\infty} \lesssim c_j 2^{\frac{j}{2}} \|a\|_{\dot H^{\frac{d-1}{2}}},$$

so thanks to the support to the Fourier transform of $T_a b$ we have

$$\begin{aligned}\big\|\Delta_\ell\big(T_a b\big)\big\|_{L^2} &\lesssim \sum_{|j-\ell|\le 5} \|S_{j-1}a\|_{L^\infty}\|\Delta_j b\|_{L^2}\\ &\lesssim c_\ell 2^{-\ell s}\|a\|_{\dot H^{\frac{d-1}{2}}}\|b\|_{\dot H^{s+\frac12}}.\end{aligned}$$

Similarly as $s>0$, it follows that

$$\begin{aligned}\big\|\Delta_\ell\big(T_b a + R(a,b)\big)\big\|_{L^2} &\lesssim \sum_{j\ge \ell-N_0} \|\Delta_j a\|_{L^2}\|S_{j+2}b\|_{L^\infty}\\ &\lesssim \sum_{j\ge \ell-N_0} c_j 2^{-js}\|a\|_{\dot H^s}\|b\|_{L^\infty} \lesssim c_\ell 2^{-\ell s}\|a\|_{\dot H^s}\|b\|_{L^\infty}.\end{aligned}$$

This achieves (A.1).

(ii) The proof is similar to that of (A.1) noticing that as $s_1 < \frac{d}{2}$,

$$\begin{aligned}\big\|\Delta_\ell\big(T_a b\big)\big\|_{L^2} &\lesssim \sum_{|j-\ell|\le 5} \|S_{j-1}a\|_{L^\infty}\|\Delta_j b\|_{L^2}\\ &\lesssim \sum_{|j-\ell|\le 5} c_j^2 2^{-j(s_1+s_2-\frac{d}{2})}\|a\|_{\dot H^{s_1}}\|b\|_{\dot H^{s_2}} \lesssim d_\ell 2^{-\ell(s_1+s_2-\frac{d}{2})}\|a\|_{\dot H^{s_1}}\|b\|_{\dot H^{s_2}}.\end{aligned}$$

The same estimate holds for $\|\Delta_\ell\big(T_b a\big)\|_{L^2}$. On the other hand, as $s_1+s_2>0$, we deduce that

$$\begin{aligned}\big\|\Delta_\ell\big(R(a,b)\big)\big\|_{L^2} &\lesssim \sum_{j\ge \ell-N_0} 2^{\frac{d}{2}\ell}\|\Delta_j a\|_{L^2}\|\widetilde{\Delta}_j b\|_{L^2}\\ &\lesssim 2^{\frac{d}{2}\ell} \sum_{j\ge \ell-N_0} c_j^2 2^{-j(s_1+s_2)}\|a\|_{\dot H^{s_1}}\|b\|_{\dot H^{s_2}} \lesssim d_\ell 2^{-\ell(s_1+s_2-\frac{d}{2})}\|a\|_{\dot H^{s_1}}\|b\|_{\dot H^{s_2}}.\end{aligned}$$

This completes the proof of (A.2). □

Appendix B. Proof of Proposition 1.7

Proposition 1.7 follows from the next statement, as the $[v_\varepsilon^{(2D)}]_\varepsilon$ part was dealt with in Lemma 2.1 of [6]. It remians prove the next result.

Proposition B.1. *Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3)$ be a divergence free vector field generating a smooth, global solution u to (NS). Then u belongs to $L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))$ and ∇u to $L^2(\mathbb{R}^+;L^\infty_v(L^2_h))$.*

Proof. We shall start by proving that u belongs to the space $L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))$. Writing

$$u = e^{t\Delta}u_0 + w \quad\text{with}\quad w \stackrel{\text{def}}{=} -\int_0^t e^{(t-t')\Delta}\mathbb{P}\mathrm{div}\,(u\otimes u)(t')\,dt',$$

we only need to prove the result for w since by the continuous embedding of $\dot H^{\frac12}(\mathbb{R}^3)$ into $\dot B^{-1}_{\infty,2}(\mathbb{R}^3)$ it is immediate to check using the definition of Besov spaces via the heat flow, that $e^{t\Delta}u_0$ belongs to $L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))$. So let us concentrate on w. By Theorems 1.1 and 2.1 of [11], u belongs to $\widetilde{L}^\infty(\mathbb{R}^+;\dot H^{\frac12}(\mathbb{R}^3))\cap\widetilde{L}^2(\mathbb{R}^+;\dot H^{\frac32}(\mathbb{R}^3))$, so we infer that u belongs to $\widetilde{L^4}(\mathbb{R}^+;\dot H^1(\mathbb{R}^3))$ and therefore $u\otimes u$ belongs to $\widetilde{L}^2(\mathbb{R}^+,\dot B^{\frac12}_{2,1}(\mathbb{R}^3))$ thanks to (A.2).

In particular there is a sequence d_ℓ in the unit sphere of $\ell^1_\ell(L^2_t)$ such that

$$\|\Delta_\ell(u\otimes u)(t)\|_{L^2}\lesssim d_\ell(t)2^{-\frac{\ell}{2}}\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot H^{\frac12})}\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot H^{\frac32})}. \tag{B.1}$$

By the Plancherel formula, we get

$$\begin{aligned}\|\Delta_\ell w(t)\|_{L^2} &\lesssim \int_0^t e^{-(t-t')2^{2\ell}}2^\ell\|\Delta_\ell(u\otimes u)(t')\|_{L^2}\,dt'\\ &\lesssim 2^{\frac{\ell}{2}}\Big(\int_0^t e^{-(t-t')2^{2\ell}}d_\ell(t')\,dt'\Big)\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot H^{\frac12})}\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot H^{\frac32})}.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we infer

$$\|\Delta_\ell w(t)\|_{L^2}\lesssim\|d_\ell(\cdot)\|_{L^2_t}2^{-\frac{3\ell}{2}}\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot H^{\frac12})}\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot H^{\frac32})}.$$

Then, using (anisotropic) Bernstein inequalities (see for instance [1]) we have

$$\|\Delta_\ell w(t)\|_{L^\infty}+\|\nabla\Delta_\ell w(t)\|_{L^2_h(L^\infty_v)}\lesssim 2^{\frac{3\ell}{2}}\|\Delta_\ell w(t)\|_{L^2}.$$

Then we conclude that

$$\sum_\ell\big(\|\Delta_\ell w\|_{L^2(\mathbb{R}^+;L^\infty)}+\|\Delta_\ell\nabla w\|_{L^2(\mathbb{R}^+;L^2_h(L^\infty))}\big)\lesssim\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+;\dot H^{\frac12})}\|u\|_{\widetilde{L}^2(\mathbb{R}^+;\dot H^{\frac32})}.$$

Let us now prove the result for $\nabla e^{t\Delta}u_0$. The proof follows the lines of the equivalence of the dyadic and heat definitions of Besov spaces (see for instance [1]). Using Lemma 2.1 of [4] and the Bernstein inequality, we get that

$$\begin{aligned}\|t^{\frac12}\nabla e^{t\Delta}u_0\|_{L^\infty_v(L^2_h)} &\le \sum_j\|t^{\frac12}2^{\frac{3j}{2}}\Delta_j e^{t\Delta}u_0\|_{L^2}\\ &\lesssim C\|u_0\|_{\dot H^{\frac12}}\sum_j t^{\frac12}2^je^{-ct2^{2j}}c_j\end{aligned}$$

where $(c_j)_{j\in\mathbb{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of ℓ^2. Using that

$$\sup_{t>0}\sum_j t^{\frac12}2^je^{-ct2^{2j}}<\infty,$$

we infer, using the Cauchy-Schwarz inequality (in j) with the weight $2^je^{-ct2^{2j}}$,

$$\begin{aligned}\|\nabla e^{t\Delta}u_0\|^2_{L^2(\mathbb{R};L^\infty_v(L^2_h))} &= \int_0^\infty t\|\nabla e^{t\Delta}u_0\|^2_{L^\infty_v(L^2_h)}\frac{dt}{t}\\ &\lesssim \|u_0\|^2_{\dot H^{\frac12}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}t^{\frac12}2^je^{-ct2^{2j}}c_j\Big)^2\frac{dt}{t}\\ &\lesssim \|u_0\|^2_{\dot H^{\frac12}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}}t^{\frac12}2^je^{-ct2^{2j}}\Big)\Big(\sum_{j\in\mathbb{Z}}t^{\frac12}2^je^{-ct2^{2j}}c_j^2\Big)\frac{dt}{t}\\ &\lesssim \|u_0\|^2_{\dot H^{\frac12}}\int_0^\infty\sum_{j\in\mathbb{Z}}t^{\frac12}2^je^{-ct2^{2j}}c_j^2\frac{dt}{t}\cdot\end{aligned}$$

Using Fubini's theorem, we infer

$$\|\nabla e^{t\Delta}u_0\|^2_{L^2(\mathbb{R};L^\infty_v(L^2_h))} \lesssim \|u_0\|^2_{\dot H^{\frac12}} \sum_{j\in\mathbb{Z}} c_j^2 \int_0^\infty t^{\frac12} 2^j e^{-ct2^{2j}} \frac{dt}{t}$$

which gives the result. □

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